DETERMINATION OF THE DIELECTRIC CONSTANT OF A TUBULAR MATERIAL AT 3KMc/s

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ABSTRACT. The paper describes two methods for determining the resonant behaviour of cylindrical cavity with a tubular dielectric material introduced coaxially in it The first method tends to give the exact value of the dielectric constant of the material The second one, based on a perturbation theory, yields somewhat approximate results

Experiments carried out at a microwave frequency of 3 KMc/s on two tubes of Pyrex glass, show that the inaccuracy in determining the dielectric constant arising out of the approximation inherent in the perturbation theory, is, in these cases, so small as to be compatible with the inaccuracy due to experimental limitations

INTRODUCTION

The treatment on the resonant behaviour of a cylindrical cavity when partially filled with a solid dielectric rod, has been given by Horner *et al* (1946). With the help of this, an exact evaluation of the dielectric constant of the rod specimen can be made by solving the transcendental equation relating the dielectric constant and the resonant frequency of the cavity with the rod placed coaxially inside. An approximate analysis on the basis of a perturbation theory was also presented by Slater (1946), with which the dielectric constant can be measured from the change in resonant frequency of the cavity with and without the specimen. This analysis can be utilised to measure the dielectric constant of a fluid (liquid, gas or plasma) in a tubular container (Biondi and Brown, 1949); the evaluation of the dielectric constant of the contaner is not required. For an exact evaluation, however, it is necessary to determine the dielectric constant of the container.

In the present work the treatment developed by Horner *et al* (1946), as well as the approximate analysis given by Slater (1946), valid for a solid cylindrical dielectric, are extended for a lossless dielectric in the form of a tube. These methods have been used at a microwave frequency of 3 KMc/s to measure the dielectric constant of two pyrex glass tubes, subsequently to be used as plasma containers.

THEORETICAL CONSIDERATIONS

The cavity is operated in the lowest frequency mode i.e. TM_{010} mode. The boundaries of the cavity are assumed to be perfectly conducting.

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(a) Exact solution

Maxwell's equations valid for the interior of the cavity are, in cylindrical coordinates (z, r, θ), :

$$\left.\begin{aligned} j\omega\,\mu H_\theta &= \frac{\partial E_z}{\partial r} \\ (\sigma+j\omega k)E_z &= \frac{1}{r}\,\frac{\partial}{\partial r}\,(rH_\theta) \end{aligned}\right\} \quad \ldots \quad (1)$$

where μ, k and σ are the permeability, permittivity and conductivity of the homogeneous dielectric medium filling the cavity and ω is the angular frequency. All the parameters are expressed in rationalised M.K.S. units.

Solutions of the above equations for E_z and H_θ are :

$$\left.\begin{aligned} E_z &= \frac{K}{\sigma+j\omega k}\,AJ_0(Kr)e^{j\omega t} \text{ volts/metre} \\ H_\theta &= AJ_1(Kr)e^{j\omega t} \text{ amp/metre} \end{aligned}\right\} \quad \ldots \quad (2)$$

in which J_0, J_1 are the Bessel functions of the first kind, A is the constant of integration governed by the strength of excitation and the propagation constant K is given by

$$K^2 = -j\omega\mu(\sigma+j\omega k) \quad \ldots \quad (3)$$

Let the cavity contain three lossless media ($\sigma = 0$) 1, 2, and 3, having permittivities k_1, k_2, k_3 respectively, as shown in Fig. 1. and permeabilities equal to

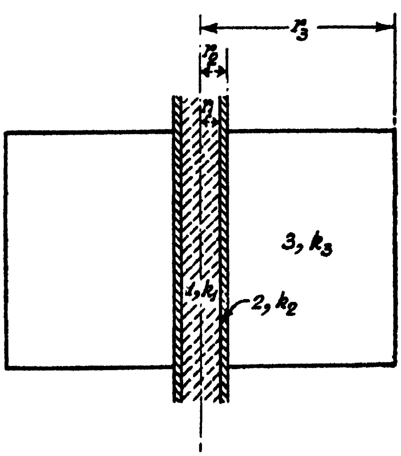

Fig 1 Cavity with glass tube

that of free space, μ_0. The propagation constants for the media are, from eqn. (3),

$$
\begin{aligned}
K_1 &= \omega\sqrt{\mu_1 k_1} = \omega\sqrt{\mu_0 k_1} = \beta_1 (\text{say}) \\
K_2 &= \omega\sqrt{\mu_2 k_2} = \omega\sqrt{\mu_0 k_2} = \beta_2 \\
K_3 &= \omega\sqrt{\mu_3 k_3} = \omega\sqrt{\mu_0 k_3} = \beta_3
\end{aligned}
\qquad \ldots \quad (4)
$$

Solutions of Maxwell's equations for the electric and magnetic fields, can be written as follows :

For medium *I*

$$
\left.
\begin{aligned}
Ez_1 &= \frac{\beta_1}{j\omega k_1} B_1 J_0(\beta_1 r) e^{j\omega t} \\
H\theta_1 &= B_1 J_1(\beta_1 r) e^{j\omega t}
\end{aligned}
\right\}
\qquad \ldots \quad (5)
$$

For medium 2

$$
\left.
\begin{aligned}
Ez_2 &= \frac{\beta_2}{j\omega k_2} [B_2 J_0(\beta_2 r) + C_2 Y_0(\beta_2 r)] e^{j\omega t} \\
H\theta_2 &= [B_2 J_1(\beta_2 r) + C_2 Y_1(\beta_2 r)] e^{j\omega t}
\end{aligned}
\right\}
\qquad \ldots \quad (6)
$$

For medium 3

$$
\left.
\begin{aligned}
Ez_3 &= \frac{\beta_3}{j\omega k_3} [B_3 J_0(\beta_3 r) + C_3 Y_0(\beta_3 r)] e^{j\omega t} \\
H\theta_3 &= [B_3 J_1(\beta_3 r) + C_3 Y_1(\beta_3 r)] e^{j\omega t}
\end{aligned}
\right\}
\qquad \ldots \quad (7)
$$

Here B's and C's are constants of integration depending upon the strength of excitation and Y_0, Y_1 are the Bessel functions of the second kind. Equation (5) does not contain the second kind Bessel functions as they become infinite at the axis of the cavity.

The boundary conditions for the cavity system are :

(1) The tangential component of the electric field at the cavity wall vanishes.

(2) There is continuity of electric and magnetic fields at the boundaries of the media 3, 2 and 2, 1. Applying these conditions we get from equations (5), (6) and (7)

$$
\begin{aligned}
& B_3J_0(\beta_3r_3)+C_3Y_0(\beta_3r_3) = 0 \\
& \frac{\beta_3}{k_3}\left[B_3J_0(\beta_3r_2)+C_3Y_0(\beta_3r_2)\right] = \frac{\beta_2}{k_2}\left[B_2J_0(\beta_2r_2)+C_2Y_0(\beta_2r_2)\right] \\
& B_3J_1(\beta_3r_2)+C_3Y_1(\beta_3r_2) = B_2J_1(\beta_2r_2)+C_2Y_1(\beta_2r_2) \qquad (8)\\
& \frac{\beta_2}{k_2}\left[B_2J_0(\beta_2r_1)+C_2Y_0(\beta_2r_1)\right] = \frac{\beta_1}{k_1}B_1J_0(\beta_1r_1) \\
& B_2J_1(\beta_2r_1)+C_2Y_1(\beta_2r_1) = B_1J_1(\beta_1r_1)
\end{aligned}
$$

Now, from equation (4)

$$\beta_1 = \beta_3\sqrt{\frac{k_1}{k_3}}$$

$$\text{and} \qquad \beta_2 = \beta_3\sqrt{\;} \qquad (9)$$

Eliminating the constants in equation (8) and using eqn. (9), we get the following transcendental equation relating the dielectric constants of the different media.

$$\left[J_0(\beta_2r_2)-\sqrt{\frac{k_2}{k_3}}FJ_1(\beta_2r_2)\right]\left[Y_0(\beta_2r_1)J_1(\beta_1r_1)-\sqrt{\frac{k_2}{k_1}}J_0(\beta_1r_1)Y_1(\beta_2r_1)\right]$$

$$=\left[\sqrt{\frac{k_2}{k_3}}FY_1(\beta_2r_2)-Y_0(\beta_2r_2)\right]\left[\sqrt{\frac{k_2}{k_1}}J_0(\beta_1r_1)J_1(\beta_2r_1)-J_0(\beta_2r_1)J_1(\beta_1r_1)\right] \ldots \quad (10)$$

$$\text{where} \qquad F = \frac{J_0(\beta_3r_2)Y_0(\beta_3r_3)-J_0(\beta_3r_3)Y_0(\beta_3r_2)}{J_1(\beta_3r_2)Y_0(\beta_3r_3)-J_0(\beta_3r_3)Y_1(\beta_3r_2)}$$

Assuming that the medium 3 is air, we get a transcendental equation from (10) which gives the dielectric constant of medium 1 in terms of that of 2 and vice versa, provided ω, the resonant frequency of the composite system is known.

Further, if we assume that the dielectric medium 1, enclosed by the tube represented by the medium 2, is air, we can find $k_2/k_0 = \epsilon_2$, the dielectric constant of the tubular material from the following equation, the permittivity of air being taken equal to that of free space, k_0.

$$\left[J_0(\beta_2r_2)-\sqrt{\frac{k_2}{k_0}}F'J_1(\beta_2r_2)\right]\left[Y_0(\beta_2r_1)J_1(\beta_0r_1)-\sqrt{\frac{k_2}{k_0}}J_0(\beta_0r_1)\,Y_1(\beta_2r_1)\right]$$

$$=\left[\sqrt{\frac{k_2}{k_0}}F'Y_1(\beta_2r_2)-Y_0(\beta_2r_2)\right]\left[\sqrt{\frac{k_2}{k_0}}J_0(\beta_0r_1)J_1(\beta_2r_1)-J_0(\beta_2r_1)J_1(\beta_0r_1)\right] \ldots \quad (11)$$

where $$F' = \frac{J_0(\beta_0 r_2)Y_0(\beta_0 r_3) - J_0(\beta_0 r_3)Y_0(\beta_0 r_2)}{J_1(\beta_0 r_2)\,Y_0(\beta_0 r_3) - J_0(\beta_0 r_3)\,Y_1(\beta_0 r_2)}$$

and $\beta_0 = w\sqrt{\mu_0 k_0} = \frac{w}{c}$, c being the velocity of light in free space.

(b) *Solution based on a Perturbation Theory*

When the electric or magnetic field within a cavity is perturbed by insertion of a material within it, a change occurs in the distribution of the electromagnetic field. Consequently the resonant frequency of the cavity changes. This change of resonant frequency, Δf, is related to the dielectric constant of the material inserted.

Let a dielectric tube be placed coaxially within the cavity. For TM_{010} mode being used, the electric field only is disturbed. Following Slater (1946), the perturbation equation is

$$\frac{\Delta f}{f_0} = -\frac{\int_{v_s}(\epsilon-1)E^2 dv}{\int_{v_c} E^2 dv} \quad \ldots \quad (12)$$

where f_0 is the resonant frequency and E is the electric field intensity of the unperturbed cavity, Δf is the change in frequency due to the introduction of the tube. v_s refers to integration over the volume of the tube and v_c that over the volume of the cavity. It should, however, be noted that this equation is valid only if the perturbation is small i.e. the dielectric constant of the material under study is not very high and the radius of the tube is small compared to that of the cavity.

Now, using cylindrical coordinates (z, r, θ), the solution of Maxwell's equation for the electric field in a cylindrical cavity operating in TM_{010} mode is, in absence of any perturbation,

$$E = DJ_0(K_0 r) \quad \ldots \quad (13)$$

where K_0 is the propagation constant and D is a constant of integration depending upon the strength of excitation.

From equations (12) and (13)

$$\frac{\Delta f}{f_0} = -\frac{1}{2}\frac{\int_z\int_r\int_\theta (\epsilon-1)D^2J_0^2(K_0 r)dr\cdot rd\theta\cdot dz}{\int_z\int_r\int_\theta D^2J_0^2(K_0 r)dr\cdot rd\theta\cdot dz}$$

$$= -\frac{1}{2}\frac{(\epsilon-1)D^2 2\pi\cdot L\int_{r_1}^{r_2} rJ_0^2(K_0 r)dr}{D^2\cdot 2\pi\cdot L\int_0^{r_3} rJ_0^2(K_0 r)dr} \quad \ldots \quad (14)$$

L represents the length of the cavity . r_1, r_2 are the internal and external radii of the tube and r_3 is the radius of the cavity (Fig. 1).

From epuation (14)

$$\frac{\Delta f}{f_0} = -\frac{1}{2}(\epsilon - 1)\frac{\int_{r_1}^{r_2} rJ_0^2(K_0 r)dr}{\int_0^{r_3} rJ^0{}_2(K_0 r)dr}$$

or,

$$\epsilon = 1 - \frac{2\Delta f}{f_0} \cdot \frac{\int_0^{r_3} rJ_0^2(K_0 r)dr}{\int_{r_1}^{r_2} rJ_0^2(K_0 r)dr}$$

$$= 1 - \frac{2\Delta f}{f_0} \cdot \frac{r_3^2[J_0^2(K_0 r_3) + J_1^2(K_0 r_3)]}{r_2^2[J_0^2(K_0 r_2) + J_1^2(K_0 r_2)] - r_1^2[J_0^2(K_0 r_1) + J_1^2(k_0 r_1)]} \quad \ldots \quad (15)$$

As the cavity wall is assumed to be perfectly conducting, the electric field at $r = r_3$ is zero. Therefore, from eauation (13), $J_0(K_0\, r_3) = 0$. Since the cavity is operated in the TM_{010} mode, $K_0 \cdot r_3 = 2.405$, the value at which the first zero of J_0 occurs.

Equation (15) becomes

$$\epsilon = 1 - \frac{2\Delta f}{f_0}$$

$$\frac{r_3^2[J_1^2(2.405)]}{r_2^2\left[J_0^2\left(2.405\frac{r_2}{r_3}\right) + J_1^2\left(2.405\frac{r_2}{r_3}\right)\right] - r_1^2\left[J_0^2\left(2.405\frac{r_1}{r_3}\right) + J_1^2\left(2.405\frac{r_1}{r_3}\right)\right]} \quad \ldots \quad (16)$$

EXPERIMENT

The block diagram of the experimental arrangement is shown in Fig. 2.

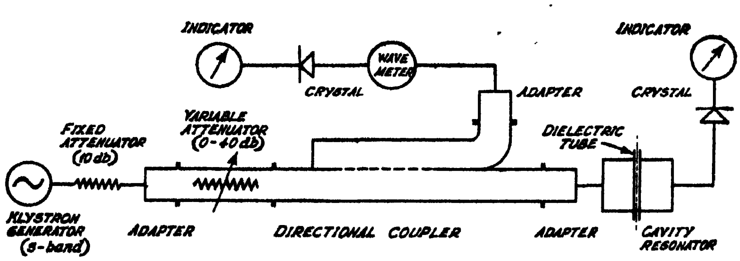

Fig 2 Experimental arrangement.

Two pyrex glass tubes were chosen as dielectric samples and each of them, in turn, was placed coaxially inside the cavity. The resonant frequencies of the cavity with and without each of the samples were measured. The resonant frequency, in each case, is given by the frequency at which the transmission through the cavity is maximum. The frequency meter gives an accurate reading within ± 0.3 Mc/sec.

RESULTS

The resonant frequency of the empty cavity in TM_{010} mode as measured experimentally (f_0') is found to be slightly different from that calculated from its dimensions (f_0). The discrepancy is assumed to be due to the presence of holes in the cavity wall provided for insertion of the samples and the coupling loops. It is therefore evident that the measured resonant frequency of the cavity with a sample (f') is also affected by the holes and needs correction, the corrected value being taken as

$$f = f' + (f_0 - f_0')$$

The frequencies f_0, f_0', f', f and the difference frequency $\Delta f = (f' - f_0')$ are given in Table I.

Radius of the cavity $r_3 = 3.837$ cm.

Sample I. Internal radius, $r_1 = 0.145$ cm
External radius, $r_2 = 0.233$ cm.

Sample II. Internal radius, $r_1 = 0.389$ cm.
External radius, $r_2 = 0.501$ cm.

The internal radii were calculated from the volume of Mercury filling the tubes as described by Worsnop and Flint (1961).

TABLE I

Sample	f_0 Mc/sec	f_0' Mc/sec	f' Mc/sec	f Mc/sec	Δf Mc/sec
I	2992.7	2999.0	2950.7	2944.4	48.3
II	2992.7	3004.5	2874.7	2862.9	129.8

Equation (11) is used for determining the exact value of the dielectric constant of the samples, puttng $\omega = 2\pi f$ where f is obtained from Table I. With the help of equation (16) based on the perturbation theory, the dielectric constant is again calculated; f_0 and Δf are given by Table I. Dielectric constant for two pyrex glass tubes as determined by the above methods, are recorded in Table II. Its percentage deviation for the solution based on the perturbation theory, with respect to the value obtained from the exact solution, is also recorded.

TABLE II

Sample	Dielectric constant from exact solution a	Dielectric constant from perturbation theory b	percentage deviation $\frac{b-a}{a} \times 100\%$
I	4.80	4.86	1.25
II	4.58	4.59	0.22

CONCLUDING REMARKS

The value of the dielectric constant of pyrex glass is found to be in conformity with that reported earlier (Von Hippel, 1954; Forsythe, 1956; Knoll, 1959). It appears from Table II that the composition of the two tube samples is slightly different.

The result for the tubes, as obtained from the solution based on the perturbation theory, differs, by less than 2%, from that given by the exact solution. It may be concluded that the inaccuracy due to the approximation inherent in the perturbation method is, in these cases, so small as to be compatible with the inaccuracy due to experimental limitations. However, if the perturbation is large, i.e., if the material under study is of high dielectric constant or if the thickness of the tubewall is appreciable, compared to the radius of the cavity resonator, the perturbation theory fails to hold. The exact method described in section (a) of theoretical consideration would, then, have to be followed.

The dielectric material discussed in this paper is assumed to be completely lossless. For a material with a low loss tangent, the methods described for determining the dielectric constant may still be applicable with a fair degree of accuracy; the loss tangent can be determined by measuring the 'Q' of the cavity with and without the specimen and applying, in an extended form, the perturbation method (Slater, 1946) or, for a more accurate evaluation, the method described by Horner *et al* (1946).

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